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Stability Analysis of Power Systems using Network Decomposition and Local Gain Analysis

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Abstract

We introduce a method for analyzing large-scale power systems by decomposing them into coupled lower order subsystems. This reduces the computational complexity of the analysis and enables us to scale the Sum of Squares programming framework for nonlinear system analysis. The method constructs subsystem Lyapunov functions which are used to estimate the region of attraction pertaining to the equilibrium point of each isolated subsystem. Then a disturbance analysis framework uses the level sets defined by these Lyapunov functions to calculate the stability of pairwise interacting subsystems. This analysis is then used to infer the stability of the entire system when an external disturbance is applied. We demonstrate the application of these techniques to the transient stability analysis of power systems.

Introduction

Direct methods for transient stability analysis of power systems avoid the expensive time-domain integration of the post-fault system dynamics. These methods rely on the estimation of the stability domain of the post-fault equilibrium point. If the initial state of the post-fault system lies inside this stability domain, then we can assert without numerically integrating the post-fault trajectory that the system will eventually converge to its post-fault equilibrium point. This inference is made by comparing the value of a carefully chosen scalar state function (energy and Lyapunov functions) at the clearing time to a critical value. In practice, finding analytical Lyapunov functions for transient stability analysis has encountered significant difficulties due to the lack of a systematic methodology for constructing a Lyapunov function (see [18]–[20] for details and a systematic survey of Lyapunov functions in power system stability). Moreover, the energy function approach suffers from the fact that energy functions for power systems with transfer conductances do not exist [8], [9]. Thus, for systems with losses, no analytical expressions are available for the estimated stability boundary of the operating point.

In [10] we have introduced an algorithm that uses Sum of Squares (SOS) techniques for the construction of Lyapunov functions for classical power system models. The proposed algorithm exploits recent system analysis methods that have opened the path toward the algorithmic analysis of nonlinear systems using Lyapunov methods [7], [11]–[16]. In order to apply these methods to power grid systems described by trigonometric nonlinearities we used an algebraic reformulation technique to recast the system's dynamics into a set of polynomial differential algebraic equations. The algorithm was embedded in an optimization loop that seeks to maximize the estimate of the Region of Attraction (ROA) of the stable operating point. This algorithm provides mathematical guarantees and avoids the major computational difficulties that are present in the energy function method. Moreover, in [10] we have also shown that systems with transfer conductances can be analyzed as well, without any conceptual difficulties.

Nevertheless, there are serious difficulties before these algebraic methods can be applied to large power systems. The difficulties are not conceptual but numerical because one of the major limitations of the SOS framework is the complexity of the system description that can currently be analyzed. It is currently very hard to construct Lyapunov functions of systems with state dimension bigger than 6, for cubic vector fields and quartic Lyapunov functions. This is a serious limitation, which renders the proposed algorithm impractical in its current formulation, as many systems of interest are of significantly higher dimension.

However, some of these numerical problems can be partially overcome by using decomposition techniques that have been proposed for the analysis of large-scale systems that are considered to be a network of lower order subsystems — see for example [1]–[7] and the references therein. Here we propose another scalable computational analysis technique for interrogating the stability properties of large-scale nonlinear dynamical systems. We employ the same decomposition technique introduced in [7] in order to derive a collection of low-order, weakly interacting subsystems. However, in order to infer the global stability analysis of the full system we propose a disturbance analysis technique that scales better than the composite Lyapunov function approach introduced in [7].

Problem Formulation and Background

Notation

The notation used is as follows. $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}_+$ denote the set of nonnegative integers. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm. The $\mathcal{L}_2$-norm of a signal $y(\cdot)$ is denoted by $\|y\|_{\mathcal{L}_2}$. The set of $n \times m$ matrices is represented by $\mathbb{R}^{n\times m}$. A matrix $P \in \mathbb{R}^{n\times n}$ is positive definite if $x^T P x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$ and positive semidefinite if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$; we denote these matrices by $P \succ 0$ and $P \succeq 0$ respectively. A monomial m_α in n independent real variables $x \in \mathbb{R}^n$ is a function of the form $m_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{Z}_+$, and the degree of the monomial is $\deg m_\alpha := \alpha_1 + \ldots + \alpha_n$. Given $c \in \mathbb{R}^k$ and $\alpha \in \mathbb{Z}_+^k$ a polynomial is defined as $p(x) = \sum_{j=1}^k c_j m_{\alpha_j}$. The degree of p is defined by $\deg p := \max_j(\deg m_{\alpha_j})$. We will denote the set of polynomials in n variables with real coefficients as $\mathcal{R}_n$ and the subset of polynomials in n variables that have maximum degree d as $\mathcal{R}_{n,d}$.

Systems analysis using sum of squares methods

A multivariate polynomial $p(x) \in \mathcal{R}_n$ is a sum of squares (SOS) if there exist some polynomial functions $h_i(x), i = 1 \ldots r$ such that

$$p(x) = \sum_{i=1}^{r} h_i^2(x)\,. \tag{1}$$

Note that $p(x)$ being a SOS implies that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$, but the converse is not always true. The set of all SOS polynomials in n variables will be denoted as Σ_n and we define $\Sigma_{n,d} = \Sigma_n \bigcap \mathcal{R}_{n,d}$. Equivalently, a polynomial $p(x) \in \mathcal{R}_n$ of degree $2d$ is a SOS if and only if there exists a positive semidefinite matrix Q and a vector of monomials $Z_{n,d}(x)$ in n variables of degree less than or equal to d such that [11]:

$$p(x) = Z_{n,d}(x)^T Q Z_{n,d}(x)\,. \tag{2}$$

For a given p finding a $Q \succ 0$ matrix is a semidefinite program (SDP) [21] which can be solved using the freely available MATLAB toolbox SOSTOOLS [22], [15] in conjunction with a semidefinite programming solver such as SeDuMi [23].

The SOS technique can be used to study the stability of dynamical systems using a Lyapunov function approach. Indeed, consider a dynamical system described by an autonomous set of nonlinear equations

$$\dot{x} = f(x)\,, \tag{3}$$

where $x \in \mathbb{R}^m$ is the state vector and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a locally Lipschitz function. We assume that $f(0) = 0$ and we are interested in the stability properties of the zero equilibrium point. Suppose that there exists an open set $D \in \mathbb{R}^m$ containing the equilibrium point $x = 0$ and a polynomial $V(x)$ such that $V(0) = 0$ and

$$V(x) - \phi_1(x) \in \Sigma_m\,, \quad \forall x \in D \diagdown \{0\}\,, \tag{4a}$$

$$-\dot{V}(x) - \phi_2(x) \in \Sigma_m\,, \forall x \in D \tag{4b}$$

where

$$\dot{V}(x) = (\partial V / \partial x)^T \cdot f(x) \tag{5}$$

and $\phi_i(x) = \epsilon_i \sum_{j=1}^m x_j^2$, with $\epsilon_i > 0, i = 1, 2$, was introduced to guarantee the positive definiteness of V and $-\dot{V}$. Then, $x = 0$ is an asymptotically stable equilibrium point. In the case in which both the vector field f and the Lyapunov function candidate V are polynomial, the Lyapunov conditions are essentially polynomial non-negativity conditions which can be NP hard to test [24]. However, if we replace the nonnegativity conditions by SOS conditions, then constructing and testing the Lyapunov function conditions can be done efficiently using SOSTOOLS. For examples and extensions see [12]–[16], [22].

Problem Formulation

Our goal is to develop a framework for the stability analysis of (3) when m or $\deg f$ is too large for the direct SOS analysis introduced in [10]. In [7] a solution to scaling SOS based Lyapunov function construction is to *decompose* system (3) into M subsystems:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1, u_1) \\
y_1 &= x_1 \\
u_1 &= [y_2^T, \ldots, y_M^T]^T \\
&\vdots \\
\dot{x}_M &= f_M(x_M) + g_M(x_M, u_M) \\
y_M &= x_M \\
u_M &= [y_1^T, \ldots, y_{M-1}^T]^T
\end{aligned} \tag{6}$$

where $g_i(x_i, 0) = 0$ for $i = 1, \ldots, M$ and all x_i and the elements of x have been permuted to produce a new state vector $x = [x_1^T, \ldots, x_M^T]^T$ with $x_i \in \mathbb{R}^{m_i}$, and $\sum_{i=1}^M m_i = m$. We define the set $\chi = \{x_1, \ldots, x_m\}$ and construct the subsets $\chi_i = \{x_i^1, \ldots, x_i^{m_i}\}$ (where x_i^j denotes the j^{th} element of x_i) that partition χ into M subsystems such that $\bigcup_{i=1}^M \chi_i = \chi$, $\chi_i \bigcap \chi_j = \emptyset$ for all $i, j = 1, \ldots, M$ $i \neq j$. (As discussed in [7] we can relax this restriction and admit overlapping partitions when some of the subsystems χ_i are not independently stable.) The objective of the decomposition is to find a partition as described in (6) that allows us to verify the stability of the overall system from subsystem Lyapunov functions. In this context a function $V_i(x_i)$ is a Lyapunov function for subsystem i if $V_i(x_i) > 0$, $dV_i/dT = \partial V_i / \partial x_i f_i(x_i) < 0, \forall x_i \neq 0$ and $V_i(0) = 0$. Unlike [7], where the subsystem Lyapunov functions V_i are used for the construction of a composite Lyapunov function, in this paper we propose a local gain analysis framework which analyzes the stability of all subsystem pairs (i, j) in order to infer the stability of system (3). The system decomposition is performed using ideas from algebraic graph theory.

Algebraic Graph Theory and Graph Partitioning

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices (nodes) $\mathcal{V} = \{v_1, \ldots, v_m\}$, and a set of edges (links) $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$. If $v_i, v_j \in \mathcal{V}$ and $e_{ij} = (v_i, v_j) \in \mathcal{E}$, then there is an edge (a directed arrow) from node v_i to node v_j . In this work we consider undirected graphs, in which case if $e_{ij} \in \mathcal{E}$ then so is e_{ji}. The graph *adjacency* matrix $A(\mathcal{G}) \in \mathbb{R}^{m \times m}$, is given by $A_{ij} = 1$ if $e_{ij} \in \mathcal{E}$ and $A_{ij} = 0$ otherwise. If $e_{ij} \in \mathcal{E}$, then nodes i and j are called neighbors. The graph *incidence* matrix $C(\mathcal{G}) \in \mathbb{R}^{m \times m}$, is given by $C_{ij} = 1$ if edge j enters v_i, -1 if edge j leaves v_i and 0 otherwise. The number of neighbours of agent i, also called the degree of vertex v_i, is denoted by n_i. The diagonal degree matrix is $D(\mathcal{G}) = \mathrm{diag}(n_i)$. The Laplacian matrix of the graph is defined as $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. For undirected graphs, $A(\mathcal{G}) = A(\mathcal{G})^T$ and $L(\mathcal{G}) \succeq 0$. Also, $L\mathbf{1} = 0$, where $\mathbf{1}$ is the n-dimensional vector of ones. For undirected graphs, the algebraic multiplicity of the zero eigenvalue of L is equal to the number of connected components in the graph. The smallest nonzero eigenvalue of the Laplacian (the Fiedler eigenvalue) is denoted by $\lambda_F(L)$ and its eigenvector is central to the proposed system decomposition algorithm.

Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the partitioning problem requires one to construct M subgraphs, $\mathcal{G}_k(\mathcal{V}_k, \mathcal{E}_k)$, $k = 1, \dots, M$ such that $\bigcup_{k=1}^{M} \mathcal{V}_k = \mathcal{V}$ and $\mathcal{V}_k \bigcap \mathcal{V}_l = \emptyset$ for all $k \neq l$ and $\mathcal{E}_k = \{(v_i, v_j) \in \mathcal{E} | v_i, v_j \in \mathcal{V}_k\}$ where the objective is to minimize the sum of the weights of the edges connecting nodes in different partitions. Partitioning a graph into two subgraphs (bisection) can be formulated as the following combinatorial optimization problem:

$$\begin{aligned} \min_z \quad & \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{m} A_{ij}(1 - z_i z_j) = \frac{1}{2} z^T L z \\ \text{s.t.} \quad & z_i^2 = 1 \quad i = 1, \dots, m \\ & z^T \mathbf{1} \neq \pm m \end{aligned} \tag{7}$$

Here $z_i = 1$ if node i is in one partition and $z_i = -1$ if node v_i is in the other. The spectral partitioning algorithm [25] approximates (7) by dropping the constraints. Each vertex is then assigned to a partition according to the rule $z_i = \operatorname{sign}(y_i)$ where y_i is the i^{th} element of the eigenvector corresponding to $\lambda_F(L)$. For fixed size partitions one can sort y into ascending order and form a partition at the median. In order to obtain multiple partitions the algorithm is recursively called on the subgraphs produced by the previous decomposition.

Dynamical System Decomposition

In [7] we have introduced a system decomposition algorithm that partition the state vector so that it minimizes the worst-case energy flow between nodes in different subsystems. The algorithm uses the linearization of the system dynamics around the equilibrium point of interest in order to define a graphical representation for the dynamical system. Thus for the nonlinear system (3) a linearization is performed around the origin such that

$$\dot{x} = F(x)\,, \tag{8}$$

where $x \in \mathbb{R}^m$ is the state vector and $F \in \mathbb{R}^{m \times m}$ is the system Jacobian matrix computed at the equilibrium point, i.e. $F = \frac{\partial f}{\partial x}|_{x=0}$.

A graphical representation of (8) can now be constructed using the adjacency matrix $A_{ij} = 1$ if $F_{ij} \neq 0$, $i \neq j$ and 0 otherwise. The decomposition algorithm (7) will produce a partition of the system into two subsystems, $S(\chi_1)$ and $S(\chi_2)$. This corresponds to a decomposition of (8) into

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{9}$$

with matrices $F_{ii} \in \mathbb{R}^{m_i \times m_i}$ and $F_{ij} \in \mathbb{R}^{m_i \times m_j}$. A desirable decomposition of (8) with $\hat{F}$ Hurwitz will turn (8) into (9) such that F_{11} and F_{22} are Hurwitz and a block diagonal Lyapunov function of the form $V_c(x) = \alpha_1 x_1^T P_{11} x_1 + \alpha_2 x_2^T P_{22} x_2$ for some $\alpha_i > 0$ exists for (8). Here $P_{ii} \succ 0$ solve the Lyapunov equations $F_{ii}^T P_{ii} + P_{ii} F_{ii} + Q_i = 0$, for $i = 1, 2$ and $Q_i \succ 0$ given, while $V_i(x_i) = x_i^T P_{ii} x_i$ is a Lyapunov function for $S(\chi_i)$.

Depending on the strength of the interactions F_{ij}, $i \neq j$, between the two subsystems and the choice of Q_i, necessary and sufficient conditions for the existence of V_c are presented in [7]. The analysis in [7] demonstrates that decomposing a system based only on the magnitude of the elements of the system matrix is not a good approach, as magnitude alone does not always provide a good indication of coupling strength. For this reason [7] introduces an algorithm that takes into account both the $\|F_{ij}\|$ and $\|P_{ii}\|$ components when decomposing the system so as to minimize the worst-case energy flow between nodes in different subgraphs. The energy flow on the edges is calculated by releasing the system from the initial condition that maximizes the observable energy at the output of the system. If we define an observable $y = C(\mathcal{G})x$, defining the flow of energy along the edges of the graph, the maximum output energy of system (8) is given by $\|y\|_{\mathcal{L}_2}^2 = x_0^T P x_0$ where $P \succ 0$ solves the Lyapunov equation $F^T P + PF + C^T C = 0$, where $C = C^T(\mathcal{G})$ and x_0 is the unit vector aligned with the eigenvector corresponding to the largest eigenvalue of P. A weighted adjacency matrix $W(\mathcal{G}) \in \mathbb{R}^{m \times m}$ is constructed where the energy flow from node i to node j is given by

$$W_{ij}(\mathcal{G}) = x_0^T X^{(ij)} x_0 \quad \text{if} \quad (i,j) \in \mathcal{E}\,, \tag{10}$$

otherwise $W_{ij}(\mathcal{G}) = 0$. The dynamical system decomposition is obtained by solving

$$\min_{z_i^2 = 1} \frac{1}{4} \sum_{i=1}^{m}\sum_{j=1}^{m} (1 - z_i z_j) W_{ij}^*\,, \tag{11}$$

where the decision vector z defines the state partition as described in and $W_{ij}^* = \frac{1}{2}(W_{ij} + W_{ji})$. Optimization (11) is exactly equivalent to a graph partitioning problem and can be solved using the spectral algorithm described before.

Set Invariance under Bounded Disturbances

Assume that we have performed a decomposition of the system (3) into $M = 2$ disjoint, but interacting subsystems

$$\dot{x}_1 = f_1(x_1) + g_1(x_1, x_2) \tag{12a}$$

$$\dot{x}_2 = f_2(x_2) + g_2(x_2, x_1)\,, \tag{12b}$$

where $x_1(t) \in \mathbb{R}^{m_1}, x_2(t) \in \mathbb{R}^{m_2}$, $f_i(x_i) = 0$, $g_1(x_1, 0) = 0$, and $g_2(x_2, 0) = 0$. Let us further assume that Lyapunov functions $V_i(x_i)$ have been already found for each isolated subsystem described by

$$\dot{x}_i = f_i(x_i)\,, \tag{13}$$

for $i = 1, 2$, and that the best estimate of the ROA for each isolated subsystem is given by $\Omega^i = \{x_i \in \mathbb{R}^{m_i} \mid V_i(x_i) < 1\}$. In order to estimate the stability of the coupled subsystems we will address the following problem. Assume that under an external disturbance the coupled subsystems evolve to the state (x_1^0, x_2^0) when the disturbance is finally cleared and the system's dynamics is controlled again by (12). The transient stability question is wether the trajectory of the system will evolve back to its equilibrium state $(x_1 = 0, x_2 = 0)$ under the coupled dynamics (12).

To answer this question we propose to solve a different, but related, problem: Given a bounded set $\Omega^1_{\gamma_1} = \{x_1 \in \mathbb{R}^{m_1} \mid V_1(x_1) \leq \gamma_1, \gamma_1 \leq 1\}$ for the first subsystem, what is the largest bounded perturbation defined by the level set $\Omega^2_{\gamma_2} = \{x_2 \in \mathbb{R}^{m_2} \mid V_2(x_2) \leq \gamma_2, \gamma_2 \leq 1\}$, so that $\dot{V}_1(x_1, x_2) = \partial V_1/\partial x_1 (f_1(x_1) + g_1(x_1, x_2)) \leq 0$ on the boundary of $\Omega^1_{\gamma_1}$?

An SOS methodology has been used before [26], [27] to consider the local effect of external disturbances on polynomial systems. There, an analysis to estimate set invariance under peak bounded disturbance was considered. In that case the peak of x_2 is bounded by $\|x_2\|_\infty \leq \gamma_2$ and an algorithm that bounds the system's reachable set under disturbances with peak less than γ_2 is proposed. Here, we modify the analysis to use the level sets defined by each subsystem Lyapunov function. Indeed, we define the disturbance x_2 to be bounded by

$$\Omega^2_{\gamma_2} = \{x_2 \in \mathbb{R}^{m_2} \mid V_2(x_2) \leq \gamma_2\}\,, \tag{14}$$

with $\gamma_2 \leq 1$ and define the invariant set as

$$\Omega^1_{\gamma_1} = \{x_1 \in \mathbb{R}^{m_1} \mid V_1(x_1) \leq \gamma_1\}\,, \tag{15}$$

where $\gamma_1 \leq 1$. We know that if $\dot{V}_1(x_1, x_2) \leq 0$ on the boundary of $\Omega^1_{\gamma_1}$ for all x_2 meeting the bound, then the flow of the system from any point in $\Omega^1_{\gamma_1}$ can not ever leave $\Omega^1_{\gamma_1}$, which makes it invariant. Expressing this condition in set containment terms, then rewriting it in set emptiness form, and using the Positivstellensatz theorem, this condition becomes (see [10] for more details)

$$\begin{aligned} s_0 + s_1(\gamma_2 - V_2(x_2)) + s_2\dot{V}_1 + s_3(\gamma_2 - V_2(x_2))\dot{V}_1 \\ + \dot{V}_1^{2k} + q(V_1 - \gamma_1) = 0\,, \end{aligned} \tag{16}$$

with $k \in \mathbb{Z}_+, q \in \mathcal{R}_{m_1+m_2}$ and $s_0, s_1, s_2, s_3 \in \Sigma_{m_1+m_2}$.

Choosing $k = 1$, we can write the following SOS constraint that guarantees the invariance of $\Omega^1_{\gamma_1}$ under bounded x_2,

$$\begin{aligned} -s_1(\gamma_2 - V_2(x_2)) - s_2\dot{V}_1 - s_3(\gamma_2 - V_2(x_2))\dot{V}_1 \\ - \dot{V}_1^2 - q(V_1 - \gamma_1) \in \Sigma_{m_1+m_2}\,. \end{aligned} \tag{17}$$

A further simplification is possible if we choose $s_3 = 0$ and we set $q \to q\dot{V}_1^2$ and $s_i \to s_i\dot{V}_1^2$ for $i = 1, 2$, since this enables us to factor out $\dot{V}_1^2$ and to obtain the following condition

$$-s_1(\gamma_2 - V_2(x_2)) - s_2\dot{V}_1 - 1 - q(V_1 - \gamma_1) \in \Sigma_{m_1+m_2}\,. \tag{18}$$

Since (18) is linear in γ_2 we can search for the maximum disturbance for which the set $\Omega^1_{\gamma_1}$ is invariant, by searching over the polynomials q and the $s_i, i = 1, 2$, to maximize γ_2 subject to (18).

Solving this SOS program for $\gamma_1 \in [0, 1]$ we generate a curve $\gamma_2(\gamma_1)$ which defines for each γ_1 level set the largest γ_2 level set disturbance for which $\Omega^1_{\gamma_1}$ remains invariant. By solving a similar problem for the disturbances applied to the second system, we generate a curve $\gamma_1(\gamma_2)$ which defines for each γ_2 level set the largest γ_1 disturbance for which $\Omega^2_{\gamma_2}$ remains invariant. Using these stability curves the stability analysis for two connected systems takes the following simple form. If for $\gamma_1 = V_1(x_1^0)$ the state $x_2^0 \in \{x_2 \in \mathbb{R}^n \mid V_2(x_2) \leq \gamma_2(\gamma_1)\}$ and if for $\gamma_2 = V_2(x_2^0)$ the state $x_1^0 \in \{x_1 \in \mathbb{R}^n \mid V_1(x_1) \leq \gamma_1(\gamma_2)\}$, then the composite system is stable.

Stability of Networked Systems

For a system composed of M interacting subsystems we conjecture that checking the stability conditions for all pairs of subsystems suffices to prove stability. Assume that at the end of an external perturbation system (6) ends in the state $(x_1^0, \ldots, x_m^0)$. Does the system evolving from this state, under the dynamics described by (6), return to its equilibrium state? Conjecture: If for each pair of subsystems (i, j) with $i \neq j$, the two coupled subsystems

$$\dot{x}_i = f_i(x_i) + g_i(x_1^0, \ldots, x_i, \ldots, x_j, \ldots, x_n^0)\,,$$

and

$$\dot{x}_j = f_j(x_j) + g_j(x_1^0, \ldots, x_i, \ldots, x_j, \ldots, x_n^0)\,,$$

return to their equilibrium state, then the system (6) returns to its equilibrium state. In other words, the point $(x_1^0, \ldots, x_n^0)$ belongs to the ROA of the stable equilibrium located at the origin. For each pair of subsystems the stability question is answered by computing the stability curves discussed in the previous section.

Results

We will consider a power system consisting of n synchronous generators. Each generator is represented by a constant voltage behind a transient reactance, constant mechanical power, and its dynamics are modeled by the swing equation. The generator voltages are denoted by $E_1\angle\delta_1, \ldots, E_n\angle\delta_n$, where $\delta_1, \ldots, \delta_n$ are the generator phase angles with respect to the synchronously rotating frame. Furthermore, the loads are represented as constant, passive impedances. Thus, this model is described by the following set of nonlinear differential equations [28]

$$\dot{\delta}_i = \omega_i\,, \tag{19a}$$

$$\dot{\omega}_i = -\lambda_i\omega_i + \frac{1}{M_i}(P_{mi} - P_{ei}(\delta))\,, \tag{19b}$$

where M_i is the generator inertia constant, $\lambda_i = D_i/M_i$, where D_i is the generator damping coefficient, P_{mi} is mechanical power input, and P_{ei} is the electrical power output,

$$\begin{aligned} P_{ei}(\delta) = E_i^2 G_{ii} + \sum_{j, j\neq i} E_i E_j [B_{ij}\sin(\delta_i - \delta_j) \\ + G_{ij}\cos(\delta_i - \delta_j)]\,, \end{aligned} \tag{20}$$

where B_{ij} and G_{ij} are the line admittances and conductances.

We assume that the dynamical system has a post-fault Stable Equilibrium Point (SEP) given by $(\delta_s, \omega_s = 0)$ where δ_s is the solution of the following set of nonlinear equations,

$$P_{mi} - P_{ei}(\delta_s) = 0, \tag{21}$$

where $i = 1, \ldots, (n-1)$. Note that we work with the relative angles with respect to a reference node, for example, node n, since the solution δ_s is invariant to a uniform translation of the angles. Moreover, since the models analyzed in this paper have uniform damping $(\lambda_i = \lambda, i = 1, \ldots, n)$, we can further reduce the phase space by working with relative speeds. Thus, the phase space dimension is $m = 2n - 2$. Finally, we make the following change of variables $\delta \to \delta + \delta_s$ in (19) in order to transfer the stable equilibrium point to the origin in phase space.

The analysis tools described in this paper were employed to perform stability analysis tests on a 7 generator 26 bus power system model. This model was obtained from the IEEE 10 generator 39 bus by removing 3 generators. For this model $n = 7$ and the dimension of the state space is $m = 12$.

Recasting the Power System Dynamics

SOS programming methods cannot be directly applied to study the stability of power system models because their dynamics contain trigonometric nonlinearities and are not polynomial. For this reason a systematic methodology to recast their dynamics into a polynomial form is necessary [12], [14]. The recasting introduces a number of equality constraints restricting the states to a manifold having the original state dimension. For the classical power system model recasting is trivially achieved by a non-linear change of variables

$$\begin{aligned} z_{3i-2} &= \sin(x_{2i-1}) \\ z_{3i-1} &= 1 - \cos(x_{2i-1}) \\ z_{3i} &= x_{2i}\,, \end{aligned}$$

for $i = 1, \ldots, n-1$. Recall that we assume a model with uniform damping so that $x_{2i-1} = \delta_i - \delta_n$ and $x_{2i} = \omega_i - \omega_n$ represent the relative angles and speeds of the generators. Recasting produces a dynamical system with a larger state dimension, $z \in \mathbb{R}^M$, where $M = 3(n-1)$ for a model with uniform damping. Recasting also introduces $(n-1)$ equality constraints,

$$G_i(z) = z_{3i-2}^2 + z_{3i-1}^2 - 2z_{3i-1} = 0\,, \tag{23}$$

where $i = 1, \ldots, n-1$, which restrict the dynamics of the new system to a nonlinear manifold of dimension m in $\mathbb{R}^M$. Note that we have chosen the recasted variables in such a way that the stable equilibrium point of the original system, $x_s = 0$, is mapped to $z_s = 0$ in the recasted system space.

Dynamical System Decomposition

The spectral algorithm described before is used to produce a decomposition of the system into 3 subsystems containing 2 generators each. In order to achieve this decomposition the the algorithm was applied recursively. We first decomposed the system into a 2 and 4 generator partition, $\hat{\chi}_1 = \{4, 5\}$ and $\hat{\chi}_2 = \{1, 2, 3, 6\}$, followed by a second decomposition of the 4 generator partition into two subsystems. In the final partition the 6 generators are distributed as $\chi_1 = \{1, 3\}, \chi_2 = \{2, 6\}$, and $\chi_3 = \{4, 5\}$. Each one of the 3 subsystems is described by a set of 6 recasted dynamic variables and 2 equality constraints:

$$\begin{aligned} S_1 : \quad & z = [z_1, z_2, z_3, z_7, z_8, z_9] \\ & z_1^2 + z_2^2 - 2z_2 = 0 \qquad\qquad \text{(24a)} \\ & z_7^2 + z_8^2 - 2z_8 = 0 \\ S_2 : \quad & z = [z_4, z_5, z_6, z_{16}, z_{17}, z_{18}] \\ & z_4^2 + z_5^2 - 2z_5 = 0 \qquad\qquad \text{(24b)} \\ & z_{16}^2 + z_{17}^2 - 2z_{17} = 0 \\ S_3 : \quad & z = [z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}] \\ & z_{10}^2 + z_{11}^2 - 2z_{11} = 0 \qquad\qquad \text{(24c)} \\ & z_{13}^2 + z_{14}^2 - 2z_{14} = 0 \end{aligned}$$

Subsystem Stability analysis

The SOS methodology described in [10] was applied to estimate the region of attraction for the isolated subsystems. Each subsystem is linearly stable at its zero equilibrium point. The first estimate of the ROA was improved using the expanding interior algorithm introduced in [10]. The evolution of the ROA estimates as the expanding interior algorithm progresses is shown in Figure 1. Notice the significant improvement of the ROA estimate by comparing the innermost contour to the outermost contour that provides the final ROA estimate.

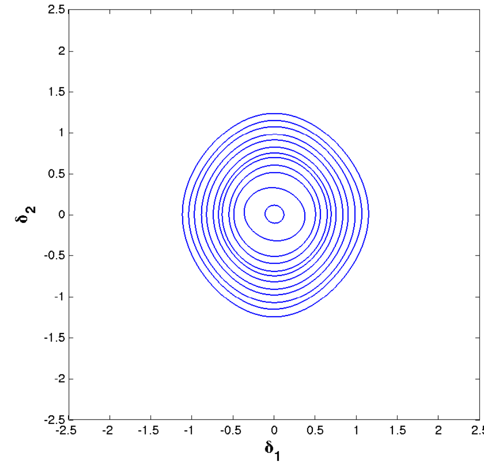

Fig. 1. Successive estimates of the ROA for subsystem 1 as projected in the angle space ($\omega_1 = \omega_2 = 0$). Each contour represents the estimate at different iterations of the expanding interior algorithm [10]. The outermost contour provides the best ROA estimate.

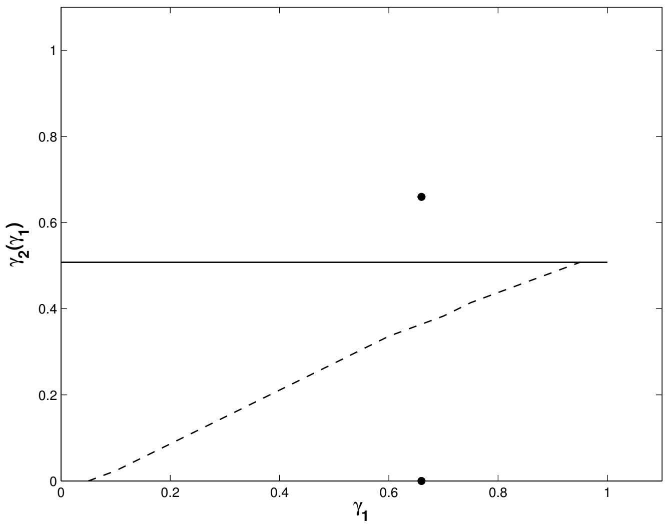

Fig. 2. Stability curves for the level sets $\Omega^1_{\gamma_1}$ of subsystem 1 under disturbances bounded by the level sets $\Omega^2_{\gamma_2}$. See text for details.

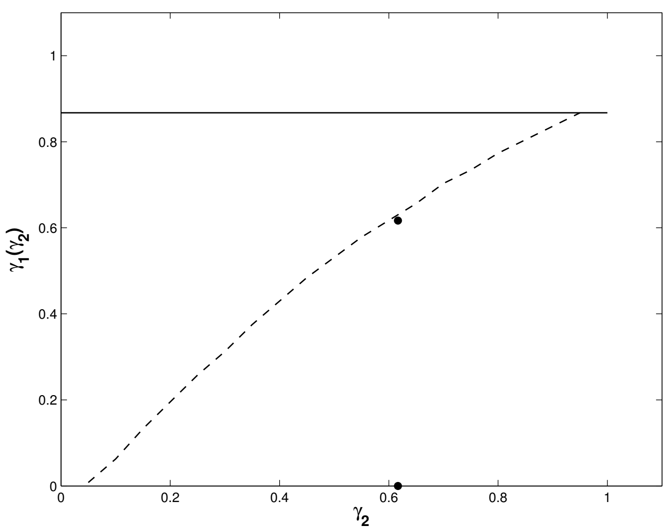

Fig. 3. Stability curves for the level sets $\Omega^2_{\gamma_2}$ of subsystem 2 under disturbances bounded by the level sets $\Omega^1_{\gamma_1}$.

Pairwise Stability Analysis

We define the following perturbation for the dynamical system

$$\begin{aligned} \chi_1 :&\quad \gamma_1 = 0.66 \quad z_0^1 = [\text{-0.67 0.26 0.01 0.52 0.15 -0.01 }] \\ \chi_2 :&\quad \gamma_2 = 0.62 \quad z_0^2 = [\text{-0.64 0.24 -0.01 0.48 0.12 0.00 }] \\ \chi_3 :&\quad \gamma_3 = 0.94 \quad z_0^3 = [\text{0.62 0.21 -0.00 0.84 0.46 -0.00 }]\,, \end{aligned}$$

where $\gamma_i = V_i(z_0^i)$. Using the Lyapunov functions for the isolated subsystems we have performed a local gain analysis for all subsystem pairs $(i, j|k)$, $i \neq j \neq k$ when the state variables of subsystem k are kept constant at z_0^k. The dashed lines in Figures 2, 3, 4 represent the $\gamma_j(\gamma_i)$ stability curves, the dots represent γ_i and γ_j values, while the continuous line marks the largest value of γ_j along this curve.

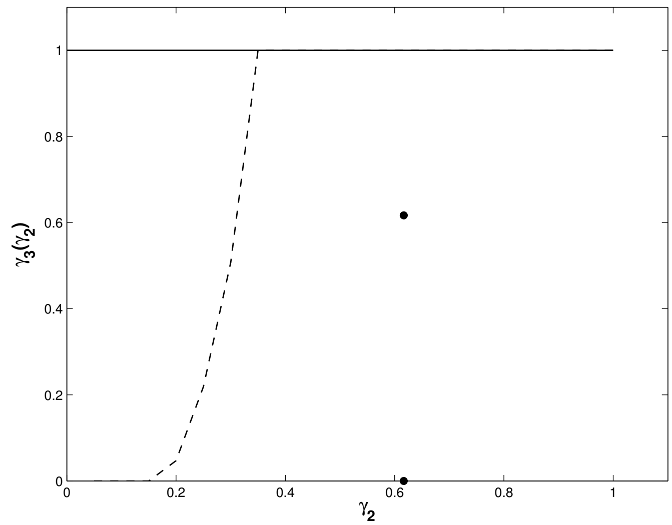

Fig. 4. Stability curves for the level sets $\Omega^2_{\gamma_2}$ of subsystem 2 under disturbances bounded by the level sets $\Omega^3_{\gamma_3}$.

For this particular disturbance all stability curves are monotonically increasing.[1] Hence, in order to decide the stability of each subsystem pair $(i, j|k)$ we have to compare $\gamma_j(\gamma_i)$ with the largest values of γ_j on this curve. If γ_j is smaller than this values, as it is the case in Figures 3 and 4 the subsystems are stable. For the other subsystem pairs $(i, j|k) = \{(2, 3|1), (1, 3|2), (3, 1|2)\}$ we found monotonically increasing stability curves with maximum γ_j values equal to 1. For the pair $(1, 2|3)$ Figure 2 shows that the largest level set Ω^1_1 is not invariant under the disturbance introduced by subsystem 2. Hence, for this disturbance we cannot guarantee that the system does not leave the set $\Omega^1_1 \times \Omega^2_1 \times \Omega^3_1$. Of course, since our analysis is conservative, the global system may actually be stable and return to its equilibrium point when this disturbance ends.

Conclusion

A method for scaling the Sum of Squares analysis framework based on dynamical system decomposition has been described. The method is based on representing a nonlinear system as a weighted graph and applying a graph partitioning algorithm to obtain the subsystems. A method for analyzing the stability of pairwise interacting subsystems using a local gain analysis method was also introduced. We have proposed a conjecture that asserts that the global stability of the system can be inferred from the pairwise stability of all its subsystems. Nevertheless this conjecture awaits a rigorous mathematical proof. We have shown how the methods described in this paper can be applied to investigate the stability of a 7 generator 26 bus system which is too large to be analyzed directly using SOS methods.

[1] We note that monotonicity is not a generic behavior of the stability curves. Moreover, for a system of interacting Van der Pol oscillators we found monotonically decreasing stability curves.

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